Math1010C Term1 2016
Tutorial 2, Sept 19

This time we evaluated $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$ by comparing it to $\lim_{n\to\infty} \frac{R^n}{n}$ for different positive R and using the $\epsilon - \mathbb{N}$ language. We also obtained the explicit formula of Fibonacci numbers by establishing some geometric sequences. For the evaluation of $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}}$, you can think that the n-th term of the sequence is taking geometric mean of $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ..., \frac{1}{n}$, so the sequence converges to 0 as the sequence $(\frac{1}{n})$ does. In Chapter XII of Lang's *Short Calculus*, Lang established two inequalities relating $n!$, n^n and the Euler's number e by considering the integral $\int_1^n log(x)dx$, which implies $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$. This of course implies $\lim_{n\to\infty} \frac{1}{\sqrt[n]{n!}} = 0$

Ex1. For $R > 0$, show that $\lim_n \dfrac{R^n}{n!} = 0$

Soln: $\dfrac{R^n}{n!} = \frac{R}{1}\frac{R}{2}...\frac{R}{n} = \frac{R}{1}\frac{R}{2}...\frac{R}{N}\frac{R}{N+1}...\frac{R}{n}$ for $n \geq N+1$ where N is any natural number. We take N to be a natural number larger than R. Then, $\dfrac{R^n}{n!} = \frac{R}{1}\frac{R}{2}...\frac{R}{n} = \frac{R}{1}\frac{R}{2}...\frac{R}{N}\frac{R}{N+1}...\frac{R}{n} \leq \frac{R}{1}\frac{R}{2}...\frac{R}{N} \cdot 1 \cdot ... \cdot 1 \cdot \frac{R}{n} = \frac{R}{1}\frac{R}{2}...\frac{R}{N}\frac{R}{n}$ for $n \geq N+1$. Since N is a fixed natural number, RHS is a constant times $\frac{R}{n}$, which converges to 0. So by Sandwich theorem, we are done.

Ex2. $\lim_n \dfrac{1}{\sqrt[n]{n!}} = 0$

Soln: For $R > 0$, we have $\lim_n \frac{R^n}{n!} = 0$. Take $\epsilon = 1$, there is a natural number $N_1 \in \mathbb{N}$ depending on R such that $0 < \frac{R^n}{n!} < 1$ for all $n \geq N_1$. Taking n-th root on both sides, $0 < \frac{R}{\sqrt[n]{n!}} < 1$ for all $n \geq N_1$. Thus, for any $R > 0$, there is a natural number $N_1 \in \mathbb{N}$ such that $0 < \frac{1}{\sqrt[n]{n!}} < \frac{1}{R}$ for all $n \geq N_1$. Let any $\epsilon > 0$, take $R = \frac{1}{\epsilon}$, then we are done.

You can use the above method to conclude that $\lim_n \sqrt[n]{n} = 1$ by comparing to $\lim_n r^n n = 0$ for any $0 < r < 1$. For $\lim_n r^n n = 0$, we give a proof here.
Proof: Let $0 < r < 1$, $r = \frac{1}{1+c}$ for some $c > 0$ (Suffices to find $c > 0$ such that the equality holds). $r^n = \frac{1}{(1+c)^n}$ and $(1+c)^n = 1+nr+\frac{n(n-1)}{2}+... \geq \frac{n(n-1)}{2}$ for $n \geq 2$. Thus, $r^n n \leq \frac{2}{n(n-1)}n = \frac{2}{n-1}$ for $n \geq 2$. LHS converges to 0. The result follows from Sandwich theorem.

Fibonacci sequence is (1,1,2,3,5,8,13,21,34,55,...). Denote the n-th term by F_n, we have $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Instead of considering the Fibonacci sequence, we consider the generalized one, called (G_n), where $G_1 = 1$, $G_2 = x$ and $G_n = G_{n-1} + G_{n-2}$ for $n \geq 3$, x is any real number. G_n can be written in terms of Fibonacci sequence, namely, $G_n = xF_{n-1}+F_{n-2}$ for $n \geq 3$. On the other hand, G_n is a geometric sequence if x satisfies $1 + x = x^2$ as can be checked. Thus, $G_n = x^{n-1} = xF_{n-1} + F_{n-2}$ whenever $1 + x = x^2$ and $n \geq 3$. Notice that $1 + x = x^2$ has two distinct roots. We denote them by ϕ_1, ϕ_2. It remains to solve the system of linear equations $\phi_i^{n-1} = \phi_i F_{n-1} + F_{n-2}$ $(i = 1, 2)$ for F_{n-1}, where ϕ_i are constants and F_{n-1}, F_{n-2} are variables in the equations.